Jordan Blocks and Exponentially Decaying Higher Order Gamow States*

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Abstract

In the framework of the rigged Hilbert space, unstable quantum systems associated with first order poles of the analytically continued S-matrix can be described by Gamow vectors which are generalized vectors with exponential decay and a Breit-Wigner energy distribution. This mathematical formalism can be generalized to quasistationary systems associated with higher order poles of the S-matrix, which leads to a set of Gamow vectors of higher order with a non-exponential time evolution. One can define a state operator from the set of higher order Gamow vectors which obeys the exponential decay law. We shall discuss to what extend the requirement of an exponential time evolution determines the form of the state operator for a quasistationary microphysical system associated with a higher order pole of the S-matrix.

1 Introduction

Gamow vectors in rigged Hilbert spaces (RHS) were introduced [1] to describe resonances, since they possess all the features usually attributed to resonance states, in particular, they obey an exponential decay law and have a Breit-Wigner energy distribution. They can be constructed from the first order poles of the analytically continued S-matrix on the second Riemann sheet of the complex energy plane, and are generalized eigenvectors of a self-adjoint Hamiltonian with complex eigenvalues (energy and lifetime). Considering the S-matrix poles

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on the second Riemann sheet of higher order, e.g. order r, one can construct in the same way r Gamow vectors of orders $k = 0,\ 1,\ 2, \ldots, r-1$. These higher order Gamow vectors are Jordan vectors of degree $k+1$, [2] and they span an r-dimensional subspace of $\mathcal{M}_{z_R} \subset \Phi^\times$, where $\Phi^\times$ is the space of generalized vectors (functionals) of the Rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$. For the self-adjoint Hamiltonian H one obtains in the space $\Phi^\times$ a matrix representation which, when restricted to the subspace $\mathcal{M}_{z_R}$, form a Jordan block of degree r. [3]

Jordan block matrices for the Hamiltonian were already investigated by Katznelson [4] who based his work on the well established ideas of Horwitz [5]. However, their Hamiltonians are non-self-adjoint, and their matrices are a generalization of the standard finite dimensional complex diagonalizable matrices for "effective" non-hermitian Hamiltonians [6]. This approach cannot be implemented in a quantum theoretical framework using the Hilbert space. In contrast, the Gamow-Jordan vectors that are derived from higher order S-matrix poles have, like ordinary Gamow vectors representing Breit-Wigner resonances, a natural place in the rigged Hilbert space formulation of quantum mechanics.

In section 2, we will give a brief introduction to rigged Hilbert spaces, ordinary and higher order Gamow vectors, and their time evolution. In section 3, we will define the exponentially decaying higher order Gamow state operator which was conjectured in reference [7] and discuss its generalization. We argue that these higher order Gamow state operators are of this form because (with certain qualifications) this is the only way to reconcile them with the exponential law.

2 Resonances, Rigged Hilbert Spaces, and Gamow Vectors

Conventionally, a resonance is described by a pair of first order poles of an analytically continued S-matrix [1]. These poles lie on the second sheet of the two-sheeted Riemann surface of complex energy at the complex conjugate positions $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$, where E_R is the resonance energy and $\hbar/\Gamma$ is the lifetime of the resonance. The pole in the lower half-plane can be associated with a state that decays exponentially (for $t > 0$), while the pole in the upper half-plane can be associated with a state that grows exponentially (for $t < 0$). While in Hilbert space there are no such states that have an exponential time evolution and distinguish a direction of time, in the rigged Hilbert space (RHS) one describes such states by antilinear continuous functionals, the Gamow vectors.

The rigged Hilbert space consists of a triplet of spaces [8]:

$$\Phi \subset \mathcal{H} \subset \Phi^\times . \tag{2.1}$$

The best known example is the RHS where Φ is the space (Schwartz space) of "well-behaved" functions, i.e. functions, that have derivatives, which are all continuous, smooth, and rapidly decreasing. $\mathcal{H}$ is the space of Lebesgue square integrable functions, and $\Phi^\times$ (the space of continous antilinear functionals on Φ) is the space of tempered distributions.

In quantum theory, if one distinguishes between preparations and registrations [9], one can further specify the RHS, and is led to a pair of RHS's: one for the preparations and one for the registrations [10]. A scattering experiment can be subdivided into a preparation stage and a registration stage. The in-state ϕ^+ that evolves from the prepared in-state $\phi^{\rm in}$ outside the interaction region is determined by the preparation apparatus (the accelerator). The out-state ψ^-, detected as the "out-state" $\psi^{\rm out}$ outside the interaction region, is determined by the registration apparatus (detector). According to the physical interpretation of the RHS formulation, "real" physical entities connected with the experimental apparatuses, e.g. the ensemble $|\phi\rangle\langle\phi|$ describing the preparation apparatus (the energy distribution of the beam) or the observable $|\psi\rangle\langle\psi|$ describing the registration apparatus (the energy resolution of the detector) are described by the "well-behaved" vectors $\phi,\ \psi \in \Phi$. One denotes the space of state vectors ϕ^+ by Φ_- and the space of "observable vectors" ψ^- by Φ_+, where $\Phi = \Phi_- + \Phi_+$ and $\Phi_- \cap \Phi_+ \neq 0$. Φ_- is the space of "well-behaved" Hardy class vectors from below and Φ_+ is the space of "well-behaved" Hardy class vectors from above, i.e. they fulfill even stronger properties connected with their analytic continuation, than the elements of the Schwartz space [11]. We will call these elements "*very* well-behaved" vectors. In place of the single rigged Hilbert space we therefore have a pair of rigged Hilbert spaces:

$$\phi^+ \in \quad \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \qquad \text{for ensembles or prepared in-states,} \tag{2.2}$$

$$\psi^- \in \quad \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \qquad \text{for observables or registered "out-states".} \tag{2.3}$$

Here the Hilbert space $\mathcal{H}$ is the same for both triplets.

On the level of Φ or $\mathcal{H}$ one cannot talk of single microsystems, and there are no mathematical objects in Hilbert space quantum mechanics to describe a single microsystem or a single experiment which prepares and observes a single microsystem. Still, it is intuitively attractive to imagine that the effect by which the preparation apparatus acts on the registration apparatus is carried out by single physical entities, the microphysical systems. The energy distribution for a microphysical system does not have to be a "well-behaved" function of the physical values of energy E. Hence, for the hypothetical entities connected with microphysical systems, like Dirac's "scattering states" $|\mathbf{p}\rangle$ or Gamow's "decaying states" $|E - i\Gamma/2\rangle$, the RHS formulation uses elements of $\Phi^\times$, $\Phi_+^\times$, and $\Phi_-^\times$.

The decaying Gamow vector associated with the complex energy $z_R = E_R - i\Gamma/2$ is a continuous antilinear functional over the vectors $\psi^- \in \Phi_+$, and its time evolution is defined for times $t \geq 0$. They are generalized eigenvectors of (the extension of) a self-adjoint

Hamiltonian H with complex eigenvalue z_R

$$H^{\times}|z_R^-\rangle = z_R|z_R^-\rangle\,; \qquad z_R = E_R - i\Gamma/2 \tag{2.4}$$

(where the $^{\times}$ denotes the conjugate operator acting on the functionals). These Gamow vectors can be obtained from the first order poles of the analytically continued S-matrix at the position z_R on the lower half-plane of the second Riemann sheet,

$$S(z) = \frac{a_{-1}}{z - z_R} + \quad \text{analytic terms}\,. \tag{2.5}$$

Here a_{-1} can determined from the unitarity of the S-matrix to be $-i\Gamma$ [12]. In the same way one can define the exponentially growing Gamow vector associated with the complex energy $z_R^* = E_R + i\Gamma/2$, which is a continuous antilinear functionals over the vectors $\phi^+ \in \Phi_-$.

The n^{th} order Gamow vector associated with the complex energy z_R is a generalization of the Gamow vector above in the following way:

1) There are r generalized vectors (functionals) of order $n = 0,\ 1,\ \ldots,\ r-1$. They are associated with a pole of the order r at the position z_R on the second Riemann sheet of the analytically continued S-matrix,

$$S(z) = \frac{a_{-1}}{z - z_R} + \frac{a_{-2}}{(z - z_R)^2} + \cdots + \frac{a_{-r}}{(z - z_R)^r} + \text{analytic terms} \tag{2.6}$$

where the coefficients $a_{-n-1},\ n = 0,\ 1, \ldots,\ r-1$ can be determined from the unitarity of the S-matrix [7] and are complex numbers with dimension $[\text{energy}]^{n+1}$.

One starts from the S-matrix elements, i.e. the matrix elements of the incoming prepared state $\phi^{\text{in}} \in \Phi_-$ at $t \to -\infty$ and the outgoing detected observable $\psi^{\text{out}} \in \Phi_+$ at $t \to \infty$

$$\begin{aligned}(\psi^{\text{out}}, S\phi^{\text{in}}) &= (\psi^{\text{out}}, \Omega^{-\dagger}\Omega^+\phi^{\text{in}}) = (\Omega^-\psi^{\text{out}}, \Omega^+\phi^{\text{in}}) \\ &= (\psi^-, \phi^+) = \int_0^\infty dE\langle\psi^-|E^-\rangle S(E)\langle^+E|\phi^+\rangle\ ,\end{aligned} \tag{2.7}$$

where $\Omega^{\pm}$ are the Møller wave operators. One can now deform the contour of integration from the cut (positive real axis and spectrum of H), into the lower half-plane of the second sheet. Then one obtains a “background integral” term, independent of the poles, along the negative real axis of the second sheet and a residue term for the pole:

$$(\psi^-, \phi^+) = \int_{-\infty_{II}}^0 dE\langle\psi^-|E^-\rangle S(E)\langle^+E|\phi^+\rangle + \text{residue at } z_R. \tag{2.8}$$

For every residue term of the S-matrix pole at z_R one obtains:

$$\begin{aligned}
\text{residue at } z_R &= \sum_{n=0}^{r-1} \frac{-2\pi i a_{-n-1}}{n!} \frac{d^n}{dz^n} \left(\langle \psi^- | z^- \rangle \langle z^+ | \phi^+ \rangle \right) \\
&= \sum_{n=0}^{r-1} \frac{-2\pi i a_{-n-1}}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}} \langle \psi^- | z^- \rangle \, \frac{d^k}{dz^k} \langle z^+ | \phi^+ \rangle \Big|_{z=z_R} \\
&= \sum_{n=0}^{r-1} \frac{-2\pi i a_{-n-1}}{n!} \sum_{k=0}^{n} \binom{n}{k} \langle \psi^- | z_R^- \rangle^{(n-k)} \; {}^{(k)}\langle {}^+ z_R | \phi^+ \rangle
\end{aligned} \tag{2.9}$$

where $\langle \psi^- | z_R^- \rangle^{(n-k)}$ is the $(n-k)$-th derivative of the "well-behaved" (continuous, analytic, smooth, rapidly decreasing) function $\langle \psi^- | z^- \rangle \in \mathcal{H}^2_- \cap \mathcal{S}$ at the position z_R, and is therefore analytic in the lower half-plane, and ${}^{(k)}\langle {}^+ z_R | \phi^+ \rangle$ is the k-th derivative of the "well-behaved" analytic function $\langle {}^+ z | \phi^+ \rangle \in \mathcal{H}^2_- \cap \mathcal{S}$ at the position z_R in the lower half-plane. The S-matrix (2.6) is thus associated with a set of r generalized vectors,

$$|z_R^-\rangle^{(0)},\ |z_R^-\rangle^{(1)}, \ldots,\ |z_R^-\rangle^{(r-1)} \tag{2.10}$$

where $|z_R^-\rangle^{(0)}$ is the ordinary Gamow vector. (The same generalization can be carried out for the growing higher order Gamow vectors with complex energy z_R^* which we shall not discuss here.)

2) It can be shown that the higher order Gamow vectors are generalized eigenvectors of the self-adjoint Hamiltonian H in the following sense

$$\begin{aligned}
\langle H\psi^- | z_R^- \rangle &= \langle \psi^- | H^\times | z_R^- \rangle^{(0)} = & z_R \langle \psi^- | z_R^- \rangle^{(0)} \\
\langle H\psi^- | z_R^- \rangle^{(1)} &= \langle \psi^- | H^\times | z_R^- \rangle^{(1)} = & z_R \langle \psi^- | z_R^- \rangle^{(1)} + \langle \psi^- | z_R^- \rangle^{(0)} \\
&\vdots \\
\langle H\psi^- | z_R^- \rangle^{(k)} &= \langle \psi^- | H^\times | z_R^- \rangle^{(k)} = & z_R \langle \psi^- | z_R^- \rangle^{(k)} + k \langle \psi^- | z_R^- \rangle^{(k-1)} \\
&\vdots \\
\langle H\psi^- | z_R^- \rangle^{(r-1)} &= \langle \psi^- | H^\times | z_R^- \rangle^{(r-1)} = & z_R \langle \psi^- | z_R^- \rangle^{(r-1)} + (r-1) \langle \psi^- | z_R^- \rangle^{(r-2)}
\end{aligned} \tag{2.11}$$

Omitting the arbitrary ψ^- one writes this as

$$H^\times |z_R^-\rangle^{(k)} = z_R |z_R^-\rangle^{(k)} + k |z_R^-\rangle^{(k-1)} \,. \tag{2.12}$$

From this it follows that $H^\times$ is a Jordan operator of degree r, and the k-th order Gamow vector $|z_R^-\rangle^{(k)}$ is a Jordan vector of degree $k+1$ [3], i.e.,

$$(H^\times - z_R)^{(k+1)}|z_R^-\rangle^{(k)} = 0 \quad \text{and} \quad (H^\times - z_R)^{(k+1)}|z_R^-\rangle^{(k-1)} \neq 0\,. \tag{2.13}$$

3) The time evolution of the decaying k-th order Gamow vector is given by [2]

$$e^{-iH^\times t}|z_R^-\rangle^{(k)} = e^{-iz_R t}\sum_{p=0}^{k}\binom{k}{p}(-it)^{k-p}\,|z_R^-\rangle^{(p)}, \qquad t \geq 0. \tag{2.14}$$

The k-th order Gamow vector evolves into a superposition containing Gamow vectors of the same and all lower orders and the space spanned by the set (2.10) is thus invariant under the action of the time evolution operator. It follows from (2.14) that the time evolution of the dyadic product of a k-th order Gamow vector with an m-th order Gamow vector, $|z_R^-\rangle^{(k)}\ {}^{(m)}\langle^- z_R|$, has terms with additional powers of t multiplying the overall exponential factor $e^{-\Gamma t}$. Such non-exponential time evolution is really no surprise in view of the previous results [14]. A more surprising fact is that certain linear combinations of the dyadic products, e.g.,

$$W^{(n)} \equiv \frac{\Gamma^n}{n!}\sum_{k=0}^{n}\binom{n}{k}|z_R^-\rangle^{(k)}\ {}^{(n-k)}\langle z_R|, \qquad \text{for fixed } n \in \{0,1,\cdots,r-1\}, \tag{2.15}$$

have a purely exponential time evolution [7]:

$$W^{(n)}(t) = e^{-iH^\times t}W^{(n)}(0)e^{iHt} = e^{-\Gamma t}W^{(n)}(0), \qquad \text{for } t \geq 0. \tag{2.16}$$

3 Form of the Exponentially Decaying Operators

We now wish to determine to what extent the requirement of exponential time evolution restricts the form of an operator constructed as a linear combination of dyadic products of vectors in $\mathcal{M}_{z_R}$. The most general linear combination of dyadic products of vectors in $\mathcal{M}_{z_R}$ is given by

$$W = \sum_{k=0}^{r-1}\sum_{m=0}^{r-1} B_{m,k}|z_R^-\rangle^{(k)}\ {}^{(m)}\langle^- z_R| \tag{3.1}$$

with arbitrary coefficients $B_{m,k}$. We will show that W decays according to the pure exponential $e^{-\Gamma t}$ if and only if the coefficients are restricted by

$$B_{m,k} = \begin{cases} \binom{k+m}{k} B_{k+m,0}, & \text{for } k+m \leq r-1, \\ 0, & \text{for } k+m > r-1, \end{cases} \tag{3.2}$$

where the coefficients $B_{k+m,0}$ remain arbitrary, that is, if and only if W is restricted to the form

$$W = \sum_{n=0}^{r-1} B_{n,0} \sum_{k=0}^{n} \binom{n}{k} |z_R^-\rangle^{(k)}\ {}^{(n-k)}\langle^- z_R| \tag{3.3}$$

with arbitrary coefficients $B_{n,0}$. Since the vectors $|z_R^-\rangle^{(k)}$ have the dimension $[\text{energy}]^{-\frac{1}{2}-k}$, the sums over k in (3.3) have the dimensions $[\text{energy}]^{-1-n}$ so that, if $2\pi\Gamma W$ is to be dimensionless, then the coefficients $B_{n,0}$ must have the dimensions $[\text{energy}]^n$ as in (2.15).

For the proof it is convenient to consider the most general linear combination of dyadic products $|z_R^-\rangle^{(k)}\ {}^{(m)}\langle^- z_R|$ for which the sum $n = k+m$ of the orders does not exceed a finite integer j; this is given by

$$W_{(j)} = \sum_{n=0}^{j} \sum_{k=0}^{n} A_{n,k} |z_R^-\rangle^{(k)}\ {}^{(n-k)}\langle^- z_R| \tag{3.4}$$

with arbitrary coefficients $A_{n,k}$. The operator (3.1) is obtained as a special case of the operator (3.4) by setting

$$j = 2(r-1), \qquad A_{n,k} = B_{n-k,k} = \begin{cases} 0, & \text{for} \quad n-k > r-1, \\ 0, & \text{for} \quad k > r-1. \end{cases} \tag{3.5}$$

The time dependence of $W_{(j)}$ is given, using (2.14), by

$$\begin{aligned} W_{(j)}(t) &= \left(e^{iHt}\right)^{\times} W_{(j)}(0) e^{iHt} = \sum_{n=0}^{j} \sum_{k=0}^{n} A_{n,k} \left(e^{iHt}\right)^{\times} |z_R^-\rangle^{(k)}\ {}^{(n-k)}\langle^- z_R| e^{iHt} \\ &= e^{-\Gamma t} \sum_{n=0}^{j} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{n-k} A_{n,k} \binom{k}{l} \binom{n-k}{m} (-it)^{k-l} (it)^{n-k-m} |z_R^-\rangle^{(l)}\ {}^{(m)}\langle^- z_R|. \end{aligned}$$

Changing the order of the summations,

$$\sum_{n=0}^{j} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{n-k} = \sum_{n=0}^{j} \sum_{l=0}^{n} \sum_{k=l}^{n} \sum_{m=0}^{n-k} = \sum_{l=0}^{j} \sum_{n=l}^{j} \sum_{k=l}^{n} \sum_{m=0}^{n-k} = \sum_{l=0}^{j} \sum_{n=l}^{j} \sum_{m=0}^{n-l} \sum_{k=l}^{n-m} = \sum_{l=0}^{j} \sum_{m=0}^{j-l} \sum_{n=l+m}^{j} \sum_{k=l}^{n-m},$$

allows the dyadic products, which are linearly independent operators, to be factored out of the sums over terms in which they appear as common factors:

$$
\begin{aligned}
W_{(j)}(t) &= e^{-\Gamma t} \sum_{l=0}^{j} \sum_{m=0}^{j-l} \sum_{n=l+m}^{j} \sum_{k=l}^{n-m} A_{n,k} \binom{k}{l} \binom{n-k}{m} (-it)^{k-l} (it)^{n-k-m} |z_R^-\rangle^{(l)} \; {}^{(m)}\langle^- z_R| \\
&= e^{-\Gamma t} \sum_{l=0}^{j} \sum_{m=0}^{j-l} |z_R^-\rangle^{(l)} \; {}^{(m)}\langle^- z_R| \sum_{n=l+m}^{j} \sum_{k=l}^{n-m} A_{n,k} \binom{k}{l} \binom{n-k}{m} (-it)^{k-l} (it)^{n-k-m} \\
&= e^{-\Gamma t} \sum_{l=0}^{j} \sum_{m=0}^{j-l} |z_R^-\rangle^{(l)} \; {}^{(m)}\langle^- z_R| \sum_{n=l+m}^{j} (it)^{n-m-l} \sum_{k=l}^{n-m} A_{n,k} \binom{k}{l} \binom{n-k}{m} (-1)^{k-l} .
\end{aligned}
$$

The operator $W_{(j)}(t)$ will decay according to the pure exponential $e^{-\Gamma t}$ if and only if all terms involving additional powers of t cancel. All terms involving additional powers of t will cancel if and only if the coefficients $A_{n,k}$ satisfy the conditions

$$
0 = \sum_{k=l}^{n-m} A_{n,k} \binom{k}{l} \binom{n-k}{m} (-1)^{k-l} \qquad \text{for} \qquad \begin{cases} l \in \{0, \cdots, j-1\}, \\ m \in \{0, \cdots, j-1-l\}, \\ n \in \{m+l+1, \cdots, j\}, \end{cases} \tag{3.6}
$$

The simplest of these conditions are those for which $n = m + l + 1$, i.e., those for which $m = n - l - 1$, because they are the only conditions that involve sums over only two values of k:

$$
\begin{aligned}
0 &= \sum_{k=l}^{l+1} A_{n,k} \binom{k}{l} \binom{n-k}{n-l-1} (-1)^{k-l} \\
&= A_{n,l} \binom{n-l}{n-l-1} - A_{n,l+1} \binom{l+1}{l} \qquad \text{for} \qquad \begin{cases} n \in \{1, \cdots, j\}, \\ l \in \{0, \cdots, n-1\} \end{cases}
\end{aligned}
$$

or, replacing l with $k-1$,

$$
A_{n,k-1} \binom{n-k+1}{n-k} = A_{n,k} \binom{k}{k-1} \qquad \text{for} \qquad \begin{cases} n \in \{1, \cdots, j\}, \\ k \in \{1, \cdots, n\} \end{cases}
$$

or, using the definition $\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$,

$$
A_{n,k} = \frac{(n-k+1)!(k-1)!}{(n-k)!k!} A_{n,k-1} \qquad \text{for} \qquad \begin{cases} n \in \{1, \cdots, j\}, \\ k \in \{1, \cdots, n\}. \end{cases}
$$

These conditions relate pairs of coefficients $A_{n,k}$ having the same values of n and successive values of k. For fixed $n \in \{1, 2, \cdots, j\}$, they may be used recursively to show that $A_{n,k}$ must equal $A_{n,0}$ multiplied by $\binom{n}{k}$:

$$\begin{aligned} A_{n,k} &= \left[\frac{(n-k+1)!(k-1)!}{(n-k)!k!}\right]\left[\frac{(n-k+2)!(k-2)!}{(n-k+1)!(k-1)!}\right]\cdots\left[\frac{(n-1)!1!}{(n-2)!2!}\right]\left[\frac{n!0!}{(n-1)!1!}\right]A_{n,0} \\ &= \frac{n!0!}{(n-k)!k!}A_{n,0} = \binom{n}{k}A_{n,0} \qquad \text{for} \qquad \begin{cases} n \in \{1, \cdots, j\}, \\ k \in \{1, \cdots, n\}. \end{cases} \end{aligned} \tag{3.7}$$

Substituting this result into the full set of conditions (3.6), using the identity

$$\binom{n}{k}\binom{k}{l}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{l}\binom{n-m-l}{k-l},$$

and then using the binomial formula gives

$$\begin{aligned} 0 &= A_{n,0}\sum_{k=l}^{n-m}\binom{n}{k}\binom{k}{l}\binom{n-k}{m}(-1)^{k-l} \\ &= A_{n,0}\binom{n}{m}\binom{n-m}{l}\sum_{k=l}^{n-m}\binom{n-m-l}{k-l}(-1)^{k-l} \\ &= A_{n,0}\binom{n}{m}\binom{n-m}{l}\sum_{k-l=0}^{n-m-l}\binom{n-m-l}{k-l}1^{n-m-k}(-1)^{k-l} \\ &= A_{n,0}\binom{n}{m}\binom{n-m}{l}(1-1)^{n} \\ &= A_{n,0}\binom{n}{m}\binom{n-m}{l}0^{n} \qquad \text{for} \qquad \begin{cases} l \in \{0, \cdots, j-1\}, \\ m \in \{0, \cdots, j-1-l\}, \\ n \in \{m+l+1, \cdots, j\}, \end{cases} \end{aligned}$$

which shows that the remaining conditions are automatically satisfied by (3.7) without placing any further conditions on the coefficients $A_{n,0}$. The coefficients $A_{n,0}$, for $n \in \{1, \cdots, j\}$, and also the coefficient $A_{0,0}$, remain completely arbitrary.

We conclude that a linear combination of dyadic products $|z_R^-\rangle^{(k)}\ {}^{(m)}\langle^- z_R|$ for which the sum $n = k + m$ of the orders does not exceed j decays according to the pure exponential $e^{-\Gamma t}$ if and only if it is of the form

$$\sum_{n=0}^{j} A_{n,0}\sum_{k=0}^{n}\binom{n}{k}|z_R^-\rangle^{(k)}\ {}^{(n-k)}\langle z_R| \tag{3.8}$$

with arbitrary coefficients $A_{n,0}$. Equation (3.3), i.e., (3.1) with the restrictions (3.2), follows by applying the special case (3.5) to the result (3.8). This concludes the proof.

4 Summary and Conclusion

In the conventional Hilbert space quantum mechanics, resonance states cannot be described by state vectors. Therefore the most common definition of a resonance is as a pole of the S-matrix at the complex energy $z_R = E_R - i\Gamma/2$ (actually, a pair of poles at $E_R \pm i\Gamma/2$). There is no theoretical reason to exclude poles of order higher than one, and poles of any order will lead to the typical resonance phenomena for cross-section and phase shift, in particular, the time delay and therewith formation of a quasi-stationary state [12]. But higher order poles have been scorned, because they were somehow associated with a time evolution that in addition to the exponential had also a strong (of order $\hbar/\Gamma$) polynomial time dependence [14] for which there exists no experimental evidence. In the rigged Hilbert space formulation of quantum mechanics, a vector description of resonances is possible, and Gamow vectors and Gamow states were defined from the first order pole of the S-matrix element. Their Hamiltonian time evolution was shown to be exactly exponential and given by a semigroup [13]. This procedure was generalized to an r-th order pole at $z = z_R$ which led to the definition of r Gamow vectors of order $k = 0,\ 1, \ldots,\ r-1$. [2] The Gamow vectors of order $k \geq 1$ were shown to have indeed a polynomial time evolution in addition to the exponential [2]. However, one can find a non-reducible state operator in the r-dimensional space $\mathcal{M}_{z_R}$ spanned by the r Gamow vectors, e.g. the operator (2.15), which has a purely exponential time evolution [7]. In the paper at hand we investigated the question, to what extend the requirement of a purely exponential time evolution with a lifetime $\tau = \hbar/\Gamma = \hbar/\mathrm{Im} z_R$ determines the form of the state opeerator. From rather plausible assumptions we concluded that this must be a "mixture" of operators like (2.15) with arbitrary coefficients $A_{n,0}$ as given by (3.8).

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